Physics 513, Quantum Field Theory

Homework 11

Due Thursday, 4th December 2003

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The Dirac Propagator

a) The Dirac propagator is defined as the time-ordered two point correlator

$$S_F(x-y)_{ab} = \langle 0|T\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle = \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & y^0 > x^0 \end{cases}.$$

We are to evaluate $S_F(x-y)_{ab}$ for a free Dirac field ψ.

Let us first compute this for the case when $x^0 > y^0$; the other case will follow trivially from symmetry arguments. Dropping all obviously zero terms, we may immediately write that

$$S_F(x-y)_{ab} = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p'}}}}e^{i(p'y-px)}\sum_{\text{spin}}\left(a^s_{\mathbf{p}}a^{s'\dagger}_{\mathbf{p'}}u^s_a(p)\bar{u}^{s'}_b(p')\right)|0\rangle.$$

Now, we know that $\langle 0|a^s_{\mathbf{p}}a^{s'\dagger}_{\mathbf{p'}}|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p'})\delta_{ss'}$ so

$$S_F(x-y)_{ab} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)}\sum_{\text{spin}}u^s_a(p)\bar{u}^s_b(p),$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}+m)_{ab}e^{-ip(x-y)},$$

$$\therefore S_F(x-y)_{ab} = (i\not{\partial}+m)_{ab}D(x-y) \quad |x^0 > y^0.$$

Now, we see that when $y^0 > x^0$, the propagator will involve the sum over spins of the v spinors which will give a $-(i\not{\partial}+m)$. This minus is cancelled by the minus in the definition of the two-point correlator.

$$\therefore S_F(x-y)_{ab} = (i\not{\partial}+m)_{ab}D(y-x) \quad |y^0 > x^0.$$

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b) We are to show that the Dirac propagator is a Green's function. Let us write the propagator as

$$S_F(x-y)_{ab} = \theta(x^0-y^0)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle.$$

When we act on this with $(i\not{\partial}-m)$, it is clear that much of the mess that follows can be greatly simplified by simple considerations. First, note that by the chain rule we will have to have terms where the partial acts on the Heaviside function multiplied by the correlator together with terms where the Heaviside function is multiplied by the partial acting on the correlator. The $-m$ term will come through the Heaviside functions and the net effect will be to have terms similar to $(i\not{\partial}-m)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$ which will be identically zero by Dirac's equations. The only terms left will be the partial derivatives acting on the Heaviside functions. This can be further simplified because $\partial\theta(x^0-y^0) = -\partial\theta(y^0-x^0)$. Therefore the entire operation reduces to

$$(i\not{\partial}-m)S_F(x-y)_{ab} = i\gamma^0\delta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle,$$

$$= i\gamma^0\delta(x^0-y^0)\langle 0|\{\psi_a(x),\psi^\dagger_b(y)\gamma^0\}|0\rangle,$$

$$= i(\gamma^0)^2\delta(x^0-y^0)\{\psi_a(x),\psi^\dagger_b(y)\}\langle 0|0\rangle,$$

$$= i\delta(x^0-y^0)d^{(3)}(\vec{x}-\vec{y})\delta_{ab},$$

$$\therefore (i\not{\partial}-m)S_F(x-y)_{ab} = i\delta^{(4)}(x-y)\delta_{ab}.$$

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c) We are to solve the equation for the Green's function equation by introducing the Fourier transform

$$S_C(x-y)_{ab} = \int_C \frac{d^4p}{(2\pi)^4} \tilde{S}_C(p)_{ab} e^{-ip(x-y)},$$

and express our answer in terms of the scalar propagator

$$D_C(x-y) = \int_C \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip(x-y)}.$$

This can be done in a rather straight-forward way. We will write the Green's function equation of part (b) in terms of the prescribed substitution for $S_C(x-y)_{ab}$.

$$(\not{p}-m) \int_C \frac{d^4p}{(2\pi)^4} \tilde{S}_C(p)_{ab} e^{-ip(x-y)} = i\delta^{(4)}(x-y).$$

We can of course bring $(\not{p}-m)$ inside the integral and it is clear that the only way for this identity to be true is if

$$(\not{p}-m)\tilde{S}_C(p)_{ab} = i.$$

If this is the case than the exponential will reduce to a simple Dirac delta functional multiplied by i which is precisely what we want. So

$$\begin{aligned} \tilde{S}_C(p) &= \frac{i}{(\not{p}-m)}, \\ &= \frac{i}{(\not{p}-m)} \frac{(\not{p}+m)}{(\not{p}+m)}, \\ &= \frac{i(\not{p}+m)}{p^2-m^2}, \end{aligned}$$

$$\therefore S_C(x-y)_{ab} = \int_C \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2} e^{-ip(x-y)} = (\not{p}+m)D_C(x-y).$$

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d) We are to use the identity

$$D_F(x-y) = \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x),$$

together with the relation derived in part (c) to reproduce the results of part (a).

Let us first write out our explicit formulation of the Dirac propagator.

$$S_F(x-y) = (i\not{\partial}+m)\left(\theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)\right).$$

Like before, we will use argumentation to reduce the problem rather than writing out explicit terms. When we act with the partial derivative operator on the Heaviside functions, we get a relative minus sign between the two terms and they will exactly cancel. They did not cancel in part (b) because there was already an inherent minus sign between the two terms. Now, because they begin additive, they will cancel. The net effect will be to bring our entire operator $(i\not{\partial}+m)$ inside the Heaviside functions completely. This will result in

$$S_F(x-y) = \theta(x^0-y^0)(i\not{\partial}+m)D(x-y) + \theta(y^0-x^0)(i\not{\partial}+m)D(y-x).$$

If you look at the two derived terms in part (a) they are identical to the equation above. Therefore, we nearly trivially reproduce the results of part (a).

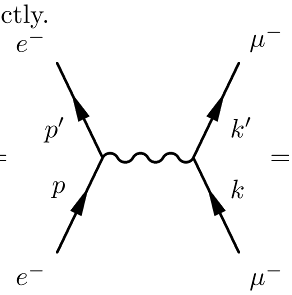
Mott's Formula (II)

In homework 9, we derived Mott's formula (the relativistic Rutherford formula). We are now to derive it by considering the spin-averaged amplitude squared of the scattering of an electron with a muon in the limit that the mass of the muon is much larger than the energy of the electron.

a) We are to compute the spin-averaged amplitude squared for $e^-\mu^-$ scattering for general m_e and m_μ.

Let us compute this directly.

$i\mathcal{M} =$



$= \frac{ie^2}{q^2}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k).$

We can compute the spin-averaged square of the amplitude directly. This becomes

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4q^4}\mathrm{Tr}\left[(\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\mathrm{Tr}\left[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\right],$$

$$= \frac{4e^4}{q^4}\left[p'^\mu p^\nu + p'^\nu p^\mu + g\mu\nu(m_e^2 - p\cdot p')\right] \times \left[k'_\mu k_\nu + k'_\nu k_\mu + g_{\mu\nu}(m_\mu^2 - k\cdot k')\right],$$

$$\therefore \overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[(p\cdot k') + (p'\cdot k)_(p\cdot k)(p'\cdot k') - m_\mu^2(p\cdot p') - m_e^2(k\cdot k') + 2m_\mu^2 m_e^2\right].$$

b) Taking the limit where m_μ is large, we can consider the case that the center of mass frame of the collision is the muon's rest frame. Therefore, we have that $k \approx k' = (m_\mu, \vec{0})$. E represents the energy of the electron. In this case, we can drastically simplify our kinematics.

$$p\cdot k = Em_\mu \qquad k\cdot k' = m_\mu^2 \qquad p\cdot p' = E^2 - \vec{p}\vec{p'} = E^2 - \vec{p}^{\,2}\cos\theta.$$

We can use this to directly write our spin-averaged squared amplitude

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[2E^2 m_\mu^2 - m_\mu^2(E^2 - \vec{p}^{\,2}\cos\theta) + m_e^2 m_\mu^2\right],$$

$$= m_\mu^2\frac{16e^4}{q^4}\left(E^2 - \vec{p}^{\,2}\sin^2\theta/2\right),$$

$$\therefore \overline{|\mathcal{M}|^2} = \frac{m_\mu^2 e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).$$

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In the last step we reduced the formula to one which will greatly help us in part (c) below.

c) We are to derive Mott's formula by taking the limit where m_μ is very large in the center of mass frame. As we stated before, this approximation is identical to assuming that the center of mass frame is actually the rest frame of the muon so our amplitude calculated in part (b) is correct to the second order. We know that the final velocity of the muon is zero and that the center of mass energy is approximately m_μ (to the first order) in this frame so we may write,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{cm}} = \frac{1}{4E_aE_b|v_a - v_b|}\frac{|\vec{p}|}{(2\pi)^2 4E_{cm}}\overline{|\mathcal{M}|^2},$$

$$= \frac{1}{4Em_\mu\beta}\frac{|\vec{p}|}{(2\pi)^2 4m_\mu}\frac{m_\mu^2 e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right),$$

$$= \frac{e^4}{16\pi^2 4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right),$$

$$\therefore \left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{cm}} = \frac{\alpha^2}{4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).$$

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